Def of eigenvalue/vector

λ eigen of A iff $A-\lambda I$ is sing.

Prop. 8.4

✓ 8.2.1 (b),

✓ 8.2.5 (a)

✓ 8.2.8

✓ 8.2.10 (a)-(c)

Prop 8.11

8.2.14

✓ 8.2.17

HW 8.2.19

✓ 8.2.20

✓ 8.2.24 (a)

HW 8.2.26 – use Thm 8.3

8.2.34(a)

8.2.35(a)

8.2.41(a)

8.2.1 (b) Find the eigenvalues and eigenvectors of

$$A = \begin{pmatrix} 1 & -2/3 \\ \frac{1}{2} & -1/6 \end{pmatrix}$$

Find characteristic eqn:

$$A - \lambda I = \begin{pmatrix} 1-\lambda & -2/3 \\ 1/2 & -1/6-\lambda \end{pmatrix}$$

$$\det(A - \lambda I) = (1-\lambda)(-1/6-\lambda) + \frac{2}{6}$$

$$= \lambda^2 - 5/6\,\lambda - \frac{1}{6} + \frac{2}{6}$$

$$= \lambda^2 - 5/6\,\lambda + \frac{1}{6}$$

$$= (\lambda - \tfrac{1}{2})(\lambda - \tfrac{1}{3}) = 0$$

$\Rightarrow$ eigenvalues : $\lambda_1 = \frac{1}{2}$, $\lambda_2 = \frac{1}{3}$

$$A - \lambda_1 I = \begin{pmatrix} 1/2 & -2/3 \\ 1/2 & -4/6 \end{pmatrix} \xrightarrow{-R1+R2} \begin{pmatrix} 1/2 & -2/3 \\ 0 & 0 \end{pmatrix}$$

$\rightarrow$ x_2 free $\quad$ $1/2\,x_1 - 2/3\,x_2 = 0$

$\Rightarrow x_1 = 4/3\,x_2$

Soln : $x_2 \begin{pmatrix} 4/3 \\ 1 \end{pmatrix}$ $\leftarrow$ eigenvectors

Same for λ_2.

8.2.5 (a) Write the char eqn for

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \alpha & \beta & \gamma \end{pmatrix}$$

$$A - \lambda I = \begin{pmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ \alpha & \beta & \gamma - \lambda \end{pmatrix}$$

$$\det(A - \lambda I) = -\lambda\left(-\lambda(\gamma - \lambda) - \beta\right) - 1(0 - \alpha) + 0$$

$$= -\lambda\left(\lambda^2 - \gamma\lambda - \beta\right) + \alpha$$

$$= -\lambda^3 + \gamma\lambda^2 + \beta\gamma + \alpha = 0$$

8.2.8 Find all the eigenval/vectors of

(a) the $n \times n$ $\underline{0}$ matrix

(b) the $n \times n$ I matrix

(a) $\underline{0}\, v = 0 = 0\, v$ for all v.

$\Rightarrow$ 0 is the only eigenvalue

all $v \neq 0$ are eigenvectors

(b) $I v = v = 1 v$ for all v

$\Rightarrow$ 1 only eigenval

all $v \neq 0$ eigenvectors.

8.2.10 Le A be an $n \times n$ matrix.

(a) Prove that if v is an eigevector of A, then cv is also an eigenvector of A, for $c \neq 0$.

Assume v is an eigenvector of A.

Then $Av = \lambda v$ for some λ.

So $A(cv) = c(Av) = c(\lambda v) = (c\lambda)v = (\lambda c)v$
$= \lambda(cv)$

$\Rightarrow$ cv is also an eigenvector. $(c \neq 0)$

(b) Prove that if v, w are eigenvectors corr. to the same egenval, λ, then $cv + dw$ is also an eigenvector $(c, d \in \mathbb{R})$.

Assume v, w eigenvect. corr. to λ.

Then $Av = \lambda v$ and $Aw = \lambda w$.

So. $A(av + bw) = A(cv) + A(dw)$
$= cAv + dAw$
$= c\lambda v + d\lambda w$
$= \lambda(cv) + \lambda(dw)$
$= \lambda(cv + dw)$

$\Rightarrow$ $cv + dw$ is eigenvector.

8.2.10 (c) Prove that if v, w are eigenvect. corr. to two diff eigenvals, λ_1, λ_2, then $cv+dw$ is not an eigenvector, $(c, d \neq 0)$.

Assume ______, then $Av = \lambda_1 v$ and $Aw = \lambda_2 w$.

If $v = kw$, then $\lambda_1 v = Av = Akw = kAw$
$= k\lambda_2 w = \lambda_2 kw$
$= \lambda_2 v$

$\Rightarrow \lambda_1 = \lambda_2$

But $\lambda_1 \neq \lambda_2$, so v and w are not mult of each $\Rightarrow$ they are lin. ind.

contradiction
suppose that $A(cv+dw) = \lambda_3 (cv+dw)$

So $A(cv + dw) = cAv + dAw = c\lambda_1 v + d\lambda_2 w$

$= \lambda_3 (cv + dw)$ iff $c\lambda_1 = c\lambda_3$
and $d\lambda_2 = d\lambda_3$

since, $c \neq 0 \Rightarrow \lambda_1 = \lambda_3$
$d \neq 0 \Rightarrow \lambda_2 = \lambda_3$ $\Rightarrow \lambda_1 = \lambda_2$

But $\lambda_1 \neq \lambda_2$

$\Rightarrow$ contr.

So $cv+dw$ is not an eigenvector.

8.2.17 Prove that the eigenvals of an upper Δ matrix, U, are its diagonal entries.

Since U is upper triang. so is $U - \lambda I$. So, $\det(U - \lambda I)$ is product of its diag entries: $\prod (u_{ii} - \lambda)$

Eigenvalues are roots of ↗ i.e. $\lambda = u_{11}, u_{22}, \ldots$ which are the diag. entries of U.

8.2.20 Show that if λ is an eigenval. of A, then λ^2 is an eigenval of A^2.

Let λ be an eigenval of A.

Then $Av = \lambda v$, for some $v \neq 0$.

So $A^2 v = A(Av) = A(\lambda v) = \lambda Av = \lambda(\lambda v) = \lambda^2 v$.

$\Rightarrow$ λ^2 is an eigenval of A^2.

8.2.24. (a) Prove that if $\lambda \neq 0$ is an eigenval of A, then $\frac{1}{\lambda}$ is an eigenval of A^{-1}

Let λ be an eigenval of A.

Then $Av = \lambda v$ for some $v \neq 0$

$\Rightarrow A^{-1}(Av) = A^{-1}(\lambda v)$

$\Rightarrow v = \lambda A^{-1} v$

$\Rightarrow \frac{1}{\lambda} v = A^{-1} v$ $\Rightarrow$ $\frac{1}{\lambda}$ eigenval. of A^{-1}.